J. Fluid Mech. (1972), *vol.* 53, *part* 4, *pp.* 637–645
Printed in Great Britain

On the calculation of wave patterns

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(Received 17 November 1971)

The calculation of the pattern of waves produced by a point disturbance in a steady field which may be non-uniform can be performed straightforwardly by a single process of integration along the characteristic rays. For illustration, the method is applied to gravity waves produced by a source moving either in a straight line or in a circular path, to the symmetrical waves produced by a source in an expanding sheet and to the waves resulting from an instantaneous disturbance in a stratified fluid.

1. Introduction

Ursell (1960) gave a method for calculating steady wave patterns that arise from constant point sources in steady non-uniform flows. He assumed, as we do here, that the waves are short compared with the length scale of variations in the basic flow. Whitham (1960), in the adjacent paper, showed that Ursell's theory can be regarded as a special case of the kinematic theory of waves. Although Whitham did not specifically consider the problem of calculating the pattern produced by a point disturbance, he applied his methods to calculating the Kelvin ship wave pattern in a later paper (Whitham 1961), though the flow is uniform in this instance. The aim of the present note is to show how straightforward the calculation of these and more general wave patterns becomes in terms of integration along characteristic rays. Oscillatory sources, instantaneous sources and the use of general co-ordinate systems are all readily handled. The need for the present simplified approach arose in an astrophysical context in which it was necessary to calculate wave patterns in self-gravitating systems that model disk galaxies. Since the purpose of this note is to illustrate the improvements inherent in the present approach, more familiar and less esoteric applications are used as illustrations.

2. The ray equations

Following Whitham, we pose the problem as that of determining the phase function $\phi(q_1, q_2; t)$ of the waves, where t is time and q_1 and q_2 are generalized spatial co-ordinates. We shall confine our attention to two-dimensional patterns, though the extension of the method to three dimensions is immediate. An equation for ϕ is normally obtained from the leading terms of an asymptotic approximation of the WKBJ type based on the assumption that the waves are short and that ϕ is large. Such an approach fulfils Ursell's desire for a theory that appears as the first stage in a well-defined scheme of successive approximations. Further, the equation for ϕ is typically a nonlinear first-order partial differential equation which we can suppose to be expressed in the form

$$\omega = W(p_i, q_i), \quad p_i = \partial\phi/\partial q_i, \quad \omega = -\partial\phi/\partial t. \tag{1}$$

The p_i become the components of a generalized wavenumber when divided by the appropriate scale factor h_i.

The standard method for such equations is to integrate along the characteristic curves given by the formulae (e.g. Garabedian 1964, p. 32)

$$dt = \frac{dq_i}{\partial W/\partial p_i} = \frac{-dp_i}{\partial W/\partial q_i} = \frac{d\omega}{0} = \frac{d\phi}{-\omega + \sum_{j=1}^{2} p_j \partial W/\partial p_j}. \tag{2}$$

When the disturbance arises from a single point source in space, the characteristic rays must all emanate from that point (Ursell's assumption A). This gives initial values for the q_i. The time t obtained from integrating equations (2) gives the travel time along a ray. As such, it is often the most convenient choice for a parameter that labels points on a particular ray. Propagation along a ray is at the group velocity $h_i^{-1}\partial W/\partial p_i$, and the requirement that t should increase along a ray is needed for determining the pattern. This requirement is equivalent to a radiation condition that the group velocity, though not necessarily the phase velocity, should always be directed away from the source (Lamb 1904). The frequency ω is seen to be constant on each ray. We shall consider both steady oscillatory sources, for which ω has some constant value ω_0 throughout the pattern, and instantaneous sources, from which rays carrying all possible values of ω originate. Ranges of initial values of the p's must also be allowed. Equation (1) applied at the source provides a single constraint on the initial values of ω and p_i. Oscillatory and instantaneous sources respectively give one- and two-parameter families of initial conditions, and hence of rays forming the pattern.

An important feature of the present approach is that the phase ϕ is also obtained by integration along the rays. Both Ursell and Whitham propose that the wave fronts be found after the rays and the wavenumber vector have been determined everywhere, by the geometric requirement that surfaces of constant phase are orthogonal to the local wavenumber vector. This typically involves a quite distinct integration from that along the rays and it is one that may be considerably more difficult, especially if the form of the dispersion relation (1) is complicated. An additional advantage of the present approach is that the value of the phase is determined at all points rather than just the forms of the surfaces of constant phase.

It can be shown that the phase obtained by integrating along the rays satisfies the geometric condition used by Ursell and Whitham; separate discussions are needed for oscillatory and instantaneous sources. To obtain the phase ϕ at any field point $A(q_1, q_2)$ at time T in the oscillatory case, we integrate from the initial value of $\phi = -\omega_0 T_1$ at the source, where $T - T_1$ is the travel time along the ray to A. It is not necessary to determine T_1 because it cancels in the expression for ϕ obtained by integrating (2):

$$\phi(A;T) = -\omega_0 T_1 - \omega_0(T-T_1) + \sum_{j=1}^{2}\int_O^A p_j \partial W/\partial p_j dt = -\omega_0 T + \sum_{j=1}^{2}\int_O^A p_j dq_j. \quad (3)$$

Here O denotes the position of the source and $\phi+\omega_0 T$ is seen to depend only on the location of A, as is to be expected. The fact that the integration is to be performed along a characteristic is most clearly expressed by writing the formal solution of (2) as $p_i = P_i(\alpha, t), q_i = Q_i(\alpha, t)$, where α is any parameter that defines an individual characteristic. Then

$$\phi(A;T)+\omega_0 T = \sum_{j=1}^{2}\int_O^A P_j(\alpha,t)\frac{\partial Q_j(\alpha,t)}{\partial t}dt. \quad (4)$$

The change in ϕ due to a small displacement in space of the field point from A to $A'(q_1+dq_1, q_2+dq_2)$ is

$$d\phi = d\alpha \sum_{j=1}^{2}\int_O^A dt\left[\frac{\partial P_j}{\partial \alpha}\frac{\partial Q_j}{\partial t}+P_j\frac{\partial^2 Q_j}{\partial \alpha\,\partial t}\right]+\sum_{j=1}^{2}p_j(A)\left[dq_j-\left(\frac{\partial Q_j}{\partial \alpha}\right)_A d\alpha\right]. \quad (5)$$

The first term arises because the characteristic from O to A' has a slightly different parameter from that to A; the second arises from that part of the displacement from A to A' not accounted for in displacement of the characteristic, and due to the possible difference in travel time from O. Integration by parts gives

$$\int_O^A P_j\frac{\partial^2 Q_j}{\partial \alpha\,\partial t}dt = \left[P_j\frac{\partial Q_j}{\partial \alpha}\right]_O^A - \int_O^A \frac{\partial P_j}{\partial t}\frac{\partial Q_j}{\partial \alpha}dt. \quad (6)$$

This expression can be simplified by using the fact that $(\partial Q_j/\partial \alpha)_O = 0$ because q_j is specified at O and so is the same for all characteristics. Furthermore, since ω is everywhere constant, we can differentiate (1) with respect to α to get

$$0 = \sum_{j=1}^{2}\left\{\frac{\partial W}{\partial p_j}\frac{\partial P_j}{\partial \alpha}+\frac{\partial W}{\partial q_j}\frac{\partial Q_j}{\partial \alpha}\right\} = \sum_{j=1}^{2}\left\{\frac{\partial Q_j}{\partial t}\frac{\partial P_j}{\partial \alpha}-\frac{\partial P_j}{\partial t}\frac{\partial Q_j}{\partial \alpha}\right\}. \quad (7)$$

The second step uses (2). When both (6) and (7) are substituted in (5), we see that

$$d\phi = \sum_{j=1}^{2} p_j(A)\,dq_j \quad (8)$$

for all spatial displacements, not just displacements along characteristics. Equation (8) shows that, on a surface of constant phase on which $d\phi = 0$, the displacement vector with components $h_j dq_j$ is perpendicular to the local wavenumber vector with components p_j/h_j.

When the source is an instantaneous one all the rays start from it at the initial instant (taken to be $t = 0$) with the same value of the phase (taken to be $\phi = 0$). At the field point A at time T,

$$\phi(A;T) = -\omega T + \sum_{j=1}^{2}\int_O^A p_j dq_j. \quad (9)$$

The rays now form a two-parameter family, for which ω can serve as one parameter; we shall use α to denote a second independent parameter. The solutions of (2) can be written as $p_i = P_i(\alpha, \omega, t), q_i = Q_i(\alpha, \omega, t)$. For a displacement of the field point to A' at fixed time T, we have

$$d\phi = -T\,d\omega + d\alpha \sum_{j=1}^{2} \int_O^A dt \left[\frac{\partial P_j}{\partial \alpha}\frac{\partial Q_j}{\partial t} + P_j \frac{\partial^2 Q_j}{\partial \alpha \partial t}\right] + d\omega \sum_{j=1}^{2} \int_O^A dt \left[\frac{\partial P_j}{\partial \omega}\frac{\partial Q_j}{\partial t} + P_j \frac{\partial^2 Q_j}{\partial \omega \partial t}\right]. \tag{10}$$

There is now no term corresponding to the second term on the right-hand side of (5) since travel times from O to A and to A' are the same. We integrate the second derivative terms by parts as before and also differentiate (1) with respect to ω to give

$$1 = \sum_{j=1}^{2} \left\{\frac{\partial W}{\partial p_j}\frac{\partial P_j}{\partial \omega} + \frac{\partial W}{\partial q_j}\frac{\partial Q_j}{\partial \omega}\right\} = \sum_{j=1}^{2} \left\{\frac{\partial Q_j}{\partial t}\frac{\partial P_j}{\partial \omega} - \frac{\partial P_j}{\partial t}\frac{\partial Q_j}{\partial \omega}\right\}. \tag{11}$$

Differentiation of (1) with respect to α also gives equation (7) because ω is independent of α. Substitution of all these formulae into (10) again gives

$$d\phi = -T\,d\omega + d\alpha \sum_{j=1}^{2} \left[P_j \frac{\partial Q_j}{\partial \alpha}\right]_A + T\,d\omega + d\omega \sum_{j=1}^{2} \left[P_j \frac{\partial Q_j}{\partial \omega}\right]_A = \sum_{j=1}^{2} p_j(A)\,dq_j, \tag{12}$$

for displacements at a fixed time.

3. Applications

The first three examples all involve steady sources ($\omega_0 = 0$) and have dispersion relations of the form

$$\omega = (\mathbf{U} + \mathbf{c}_0).\mathbf{k}, \tag{13}$$

where $\mathbf{k}$ is the wavenumber vector, $\mathbf{U}$ is the velocity of an underlying flow and $\mathbf{c}_0$ is the velocity waves would have if there were no flow.

3.1. *Kelvin's ship wave pattern*

We take axes moving with the source at the origin, so that the undisturbed flow has velocity V in the direction of the x axis. Using Cartesian co-ordinates,

$$\omega = Vk_x + (gk)^{\frac{1}{2}} = 0, \tag{14}$$

where g is the acceleration due to gravity, $k_x = p_x$, $k_y = p_y$, $k = (p_x^2 + p_y^2)^{\frac{1}{2}}$. Both k_x and k_y are constant on each characteristic since there is no explicit dependence of ω on x and y. Equations (2) give

$$dt = \frac{dx}{V + \frac{1}{2}k_x(g/k^3)^{\frac{1}{2}}} = \frac{dy}{\frac{1}{2}k_y(g/k^3)^{\frac{1}{2}}} = \frac{d\phi}{Vk_x + \frac{1}{2}(gk)^{\frac{1}{2}}}. \tag{15}$$

Integration is immediate since all the denominators are constant, and the solution is obtained simply by omitting the d's. The rays are all straight lines through O. We can introduce polar co-ordinates in wavenumber space defined by $k_x = -k\cos\chi$, $k_y = -k\sin\chi$; both k and χ are then constant on each ray but are not independent since (14) requires $V\cos\chi = (g/k)^{\frac{1}{2}}$. If we use this relation to eliminate k in (15) we obtain

$$x = \frac{-V^2\phi}{4g}(5\cos\chi - \cos 3\chi), \quad y = \frac{V^2\phi}{4g}(\sin\chi + \sin 3\chi), \tag{16}$$

which is consistent with the familiar parametric form (with parameter χ) for the wave fronts (Lamb 1932, §256). The phase ϕ, rather than t, is the variable along a ray in the integrals (16). The two quantities are linearly related for each ray with $\phi = -\frac{1}{2}Vkt\cos\chi$, which shows that only negative values of ϕ are relevant and that the pattern lies wholly downstream from the source. The parameter χ can vary in the range $-90° \leqslant \chi \leqslant 90°$. The corresponding rays are straight lines pointing downstream and the slope y/x increases from 0 at $\chi = -90°$ to $1/2\sqrt{2}$ at $\chi = -35° \, 16'$, decreases to $-1/2\sqrt{2}$ at $\chi = 35° \, 16'$ and finally increases to 0 again at $\chi = 90°$. Each direction in the wedge $-2x\sqrt{2} < y < 2x\sqrt{2}$ has two rays, one propagating the transverse waves $(-35° \, 16' < \chi < 35° \, 16')$ and the other propagating the diverging waves $(90° \geqslant |\chi| > 35° \, 16')$.

3.2. *Ship in a circular course of radius R*

It is convenient here to work in polar spatial co-ordinates with origin at the centre of the path which rotate with the source at $r = R$, $\theta = 0$. The source is supposed to be rotating anti-clockwise with angular velocity Ω. We now have a spatially non-uniform situation, for which the dispersion relation in the rotating frame is

$$\omega = -r\Omega k_\theta + g^{\frac{1}{2}}[k_r^2 + k_\theta^2]^{\frac{1}{4}} = 0. \tag{17}$$

The components of the wavenumber vector are $k_r = \partial\phi/\partial r = p_r$ and

$$k_\theta = \partial\phi/r\partial\theta = p_\theta/r,$$

and equations (2) for the characteristics give

$$dt = \frac{dr}{\frac{1}{2}p_r g^{\frac{1}{2}}[p_r^2 + p_\theta^2/r^2]^{-\frac{3}{4}}} = \frac{r^2 d\theta}{-r^2\Omega + \frac{1}{2}p_\theta g^{\frac{1}{2}}[p_r^2 + p_\theta^2/r^2]^{-\frac{3}{4}}} = \frac{dp_\theta}{0}$$

$$= \frac{r^3 dp_r}{\frac{1}{2}p_\theta^2 g^{\frac{1}{2}}[p_r^2 + p_\theta^2/r^2]^{-\frac{3}{4}}} = \frac{2d\phi}{-\Omega p_\theta}. \tag{18}$$

The phase ϕ is again always negative. The quantity p_θ, which from (17) must be positive, is constant on each ray and p_r is given by (17) as

$$p_r = \pm p_\theta \left[\frac{\Omega^4 p_\theta^2}{g^2} - \frac{1}{r^2}\right]^{\frac{1}{2}}, \tag{19}$$

with the upper or lower choice of sign depending on whether r is increasing or decreasing along the ray.

There are alternative routes to the integration of (18) and we shall employ one that gives an explicit equation for the rays, rather than a parametric one. Non-dimensional formulae that can be derived from (18) are

$$\frac{d\theta}{dr'} = \operatorname{sgn}(p_r)\frac{1 - 2\beta^2 r'^2}{r'[\beta^2 r'^2 - 1]^{\frac{1}{2}}}, \quad \frac{d\phi}{dr'} = \frac{-\operatorname{sgn}(p_r)\, g\beta^3 r'}{R\Omega^2[\beta^2 r'^2 - 1]^{\frac{1}{2}}}, \tag{20}$$

where we define $r' = r/R$ and $\beta = R\Omega^2 p_\theta/g$. The appropriate solutions of these equations are

$$\theta \operatorname{sgn}(r' - 1) = \sin^{-1}(1/\beta) - \sin^{-1}(1/\beta r') + 2(\beta^2 - 1)^{\frac{1}{2}} - 2(\beta^2 r'^2 - 1)^{\frac{1}{2}}, \tag{21}$$

$$\phi \operatorname{sgn}(r' - 1) = (g\beta/\Omega^2 R)\,[(\beta^2 - 1)^{\frac{1}{2}} - (\beta^2 r'^2 - 1)^{\frac{1}{2}}]. \tag{22}$$

Now β, which must be greater than or equal to 1, is a parameter that is constant on a characteristic. There are two characteristics for each value of β, one travelling radially outwards and the other travelling inwards. The ingoing ray is turned around and reflected smoothly at $r' = 1/\beta$, where $dr' = 0$. The integrals of (20) for the reflected rays are found after matching with the ingoing rays to be

$$\theta = \pi - \sin^{-1}(1/\beta) - \sin^{-1}(1/\beta r') - 2(\beta^2-1)^{\frac{1}{2}} - 2(\beta^2 r'^2 - 1)^{\frac{1}{2}}, \tag{23}$$

$$\phi = (-g\beta/\Omega^2 R)\,[(\beta^2-1)^{\frac{1}{2}} + (\beta^2 r'^2 - 1)^{\frac{1}{2}}]. \tag{24}$$

The present results for the wave fronts agree with those of Stoker (1957, chap. 8), the pattern being a skewed form of the ordinary Kelvin pattern. Stoker found the forms of the wave fronts by a direct application of stationary phase arguments, and his parametric equations (8.2.24) can be shown to be equivalent to ours. His equations can also be derived more directly by integrating equations (18) with respect to t, which yields

$$\left.\begin{aligned} r'^2 &= 1 + \Omega t \sin\alpha\cos\alpha + \tfrac{1}{4}\Omega^2 t^2 \cos^2\alpha, \\ \theta &= -\alpha - \Omega t + \tan^{-1}[\tfrac{1}{2}\Omega t + \tan\alpha], \\ \phi &= -\tfrac{1}{2}\Omega p_\theta t. \end{aligned}\right\} \tag{25}$$

Here $\sec\alpha = \beta =$ Stoker's θ, Stoker's $\kappa = \Omega t \sec\alpha$, and ingoing and outgoing rays are given respectively by negative and positive values of $\sin\alpha$.

The ray treatment does show some interesting new results. The outgoing rays for which $1 < \beta \leqslant (\frac{3}{2})^{\frac{1}{2}}$ form an envelope, which an individual ray touches at the point where $r' = (4\beta^2-3)^{\frac{1}{2}}/2\beta(\beta^2-1)^{\frac{1}{2}}$. This envelope or caustic forms the outer edge of the pattern, a role played by a single ray in the Kelvin pattern. The envelope is initially inclined to the flow at an angle of 19° 28′ and tangent to the ray with $\beta = (\frac{3}{2})^{\frac{1}{2}}$, but thereafter the inclination decreases steadily as the outer boundary spirals out to infinity. The rays that form the outer caustic carry transverse waves until they touch the caustic, after which they carry diverging waves, as do the rays with $\beta > (\frac{3}{2})^{\frac{1}{2}}$. These rays with $\beta > (\frac{3}{2})^{\frac{1}{2}}$ form no envelope. The extreme rays are that with $\beta = 1$, which spirals outwards, and that with $\beta = \infty$, which is the circle $r' = 1$.

The inner boundary of the pattern, on the other hand, is the envelope formed by the ingoing rays with $\beta \geqslant (\frac{3}{2})^{\frac{1}{2}}$. These carry diverging waves before they touch the caustic, and transverse waves thereafter. Individual rays touch the envelope at $r' = (4\beta^2-3)^{\frac{1}{2}}/2\beta(\beta^2-1)^{\frac{1}{2}}$. This envelope, which also is initially inclined at an angle of 19° 28′, is less steeply inclined thereafter and reaches the centre only after infinitely many circuits around it. All ingoing rays are eventually reflected, and the reflected rays form no envelope. No rays would reach the centre if surface tension were also taken into account, for then the fact that the minimum phase velocity is $2(g\tilde{T}/\rho)^{\frac{1}{4}}$ ensures that no waves can penetrate within a distance of $2(g\tilde{T}/\rho\Omega^2)^{\frac{1}{4}}$ from the centre. Here $\tilde{T}$ is the surface tension and ρ is the density.

3.3. *Symmetrical capillary waves in a radially expanding sheet*

We consider a thin sheet of thickness $h = h_0/r$, which is expanding radially, h_0 being a constant. Using Taylor's (1959) formula $k(\tilde{T}h/\rho)^{\frac{1}{2}}$ for the phase velocity of the dispersive symmetrical waves, we obtain the dispersion relation in polar co-ordinates:

$$\omega = C_0 k_r + k^2(\tilde{T}h_0/\rho r)^{\frac{1}{2}} = 0. \tag{26}$$

The characteristic equations are

$$dt = \frac{dr}{C_0 + 2p_r(\tilde{T}h_0/\rho r)^{\frac{1}{2}}} = \frac{r^2 d\theta}{2p_\theta(\tilde{T}h_0/\rho r)^{\frac{1}{2}}} = \frac{2r^3 dp_r}{(5p_\theta^2 + r^2 p_r^2)(\tilde{T}h_0/\rho r)^{\frac{1}{2}}}$$
$$= \frac{dp_\theta}{0} = \frac{d\phi}{-C_0 p_r}. \tag{27}$$

Again p_θ is constant, and (26) can be solved for p_r to give

$$p_r = \frac{-C_0}{2}\left(\frac{\rho r}{\tilde{T}h_0}\right)^{\frac{1}{2}} \pm \frac{1}{2}\left(\frac{\rho r C_0^2}{\tilde{T}h_0} - \frac{4p_\theta^2}{r^2}\right)^{\frac{1}{2}}, \tag{28}$$

where the upper or lower choice of sign is appropriate according to whether r is increasing or decreasing along the ray. We shall again derive explicit equations for the rays. Integration in terms of t in this instance introduces elliptic functions. The equations to be solved are then

$$\left.\begin{aligned} \frac{d\theta}{dr} &= \pm\frac{2p_\theta}{r^2(C_0^2\rho r/\tilde{T}h_0 - 4p_\theta^2/r^2)^{\frac{1}{2}}}, \\ \frac{d\phi}{dr} &= \frac{-C_0}{2}\left(\frac{\rho r}{\tilde{T}h_0}\right)^{\frac{1}{2}} \pm \frac{C_0^2 \rho r}{2\tilde{T}h_0(C_0^2\rho r/\tilde{T}h_0 - 4p_\theta^2/r^2)^{\frac{1}{2}}}. \end{aligned}\right\} \tag{29}$$

For definiteness, we consider the problem discussed by Taylor and Ursell of a wave source at the point $r = R$, $\theta = 0$ of the sheet. Rays leave this source in all directions and are described by

$$\frac{3|\theta|}{2}\operatorname{sgn}(r-R) = \sin^{-1}\left(\frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2 R^3}\right)^{\frac{1}{2}} - \sin^{-1}\left(\frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2 r^3}\right)^{\frac{1}{2}}, \tag{30}$$

$$\frac{3\phi}{C_0}\left(\frac{\tilde{T}h_0}{\rho}\right)^{\frac{1}{2}} = R^{\frac{3}{2}} - r^{\frac{3}{2}} + \operatorname{sgn}(r-R)\left[\left(r^3 - \frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2}\right)^{\frac{1}{2}} - \left(R^3 - \frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2}\right)^{\frac{1}{2}}\right]. \tag{31}$$

The rays for which r initially decreases are smoothly reflected at the point $r = (4\tilde{T}h_0 p_\theta^2/\rho C_0^2)^{\frac{1}{3}}$, $|\theta| = \frac{1}{3}\pi - \frac{2}{3}\sin^{-1}(4\tilde{T}h_0 p_\theta^2/\rho C_0^2 R^3)^{\frac{1}{2}}$, with the exception of the $p_\theta = 0$ ray, which travels directly into the origin. The formulae appropriate to the reflected rays are

$$\frac{3|\theta|}{2} = \pi - \sin^{-1}\left(\frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2 R^3}\right)^{\frac{1}{2}} - \sin^{-1}\left(\frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2 r^3}\right)^{\frac{1}{2}}, \tag{32}$$

$$\frac{3\phi}{C_0}\left(\frac{\tilde{T}h}{\rho_0}\right)^{\frac{1}{2}} = R^{\frac{3}{2}} - r^{\frac{3}{2}} + \left(r^3 - \frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2}\right)^{\frac{1}{2}} + \left(R^3 - \frac{4\tilde{T}h_0 p_\theta^2}{\rho C_0^2}\right)^{\frac{1}{2}}. \tag{33}$$

The rays fan out from the source and cover the whole sheet in a continuous manner and no caustic is formed. Expressions (31) and (33) for ϕ are readily shown to be equivalent to

$$\frac{3\phi}{C_0}\left(\frac{\tilde{T}h}{\rho_0}\right)^{\frac{1}{2}} = R^{\frac{3}{2}} - r^{\frac{3}{2}} + [R^3 - 2(Rr)^{\frac{3}{2}}\cos\tfrac{3}{2}\theta + r^3]^{\frac{1}{2}}, \tag{34}$$

[c.f. Ursell's equation (4.20)], which shows that ϕ increases steadily along each ray. Near the source, the curves of constant phase are locally a set of confocal parabolae with foci at the source.

3.4. *Instantaneous source in a stratified fluid*

As an example of an unsteady pattern, we consider the sudden collapse of a tubular region of mixed fluid in an otherwise vertically stratified fluid. This problem has been investigated both theoretically and experimentally by Wu (1969). If we idealize the mixed region as an instantaneous two-dimensional line source the problem becomes amenable to the present methods. We define the source line as Oz, with y axis vertical and x axis horizontal. The dispersion relation in a Boussinesq approximation is then

$$\omega = \pm Nk_x/k, \quad N^2 = -(g/\rho)\,d\rho/dy. \tag{35}$$

Here ρ is the undisturbed density and N^2 is taken to be constant for simplicity. The characteristic equations now yield

$$dt = \pm\frac{k^3 dx}{Nk_y^2} = \mp\frac{k^3 dy}{Nk_x k_y} = \frac{dk_x}{0} = \frac{dk_y}{0} = \frac{d\omega}{0} = \frac{d\phi}{-\omega}. \tag{36}$$

Both the frequency and wavenumber vector are constant on each ray, and the rays are all straight lines through O. A convenient choice of parameters for labelling the rays are polar co-ordinates in wavenumber space, defined now by $k_x = k\sin\chi$, $k_y = -k\cos\chi$. Each physically distinct wave is included twice if all possible sign combinations are allowed, but if the $\pm$ choice is replaced by $\operatorname{sgn}(\cos\chi)$ a continuous representation of all the distinct waves is achieved in $0 \leqslant \chi < 2\pi$. The solution for the wave pattern is

$$\left.\begin{aligned} x = (Nt/k)\cos\chi\,|\cos\chi|, \quad y = (Nt/k)\sin\chi\,|\cos\chi| = x\tan\chi,\\ \phi = -Nt\,|\cos\chi|\tan\chi = -Nty/x[1+y^2/x^2]^{\frac{1}{2}}. \end{aligned}\right\} \tag{37}$$

It is readily seen that each ray carries waves of all lengths and that propagation velocities increase with wavelength. The curves of constant phase at any instant are also straight lines through O, as is seen from the second expression for ϕ in (37). The x axis remains the curve $\phi = 0$ always, while new wave crests are continually formed in both the upward and downward vertical directions after successive time intervals of $2\pi/N$. They subsequently swing steadily round towards the horizontal. This general prediction is confirmed by Wu's experiments, though the finite size of the experimental source clearly introduces additional effects.

4. Comments and conclusions

The above examples are intended to show the simplicity and directness of the present method. Part of the simplicity is due to the fact that at most one spatial variable appears explicitly in any dispersion relation. The necessary integrations then reduce to quadratures, which can be performed in terms of elementary functions in the chosen examples. It is not difficult to construct similar examples in which the quadratures cannot be done in finite terms, or in which the dispersion relation is more general, and such have arisen in the author's studies. However, the present method is readily adaptable to quite general problems, for which the system of characteristic equations can be integrated numerically step by step. The phase ϕ can be used as independent variable, and this allows curves at constant and regular intervals of ϕ to be readily obtained. The construction of wave fronts is much more complicated in other methods. Caution is necessary when ϕ is used as independent variable because the phase does not necessarily change monotonically along a ray, though it does do so in all the examples of §3. No such difficulty arises when t is used as the independent variable. A further cautionary remark concerning the phase is that it may change discontinuously at a caustic, as it is known to do by an amount $\frac{1}{2}\pi$ in the ship wave problem. Strictly, a more accurate analysis than the present one is needed in the neighbourhood of the caustic.

This work has been supported in part by the National Science Foundation under grant GP-30136. It is contribution No. 60 of the Geophysical Fluid Dynamics Institute, Florida State University.

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